BIFURCATIONS OF PERIODIC ORBITS IN THE HAMILTONIAN SYSTEM WITH THREEFOLD SYMMETRY

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In a recent work the author investigated bifurcations of periodic orbits in the Hénon-Heiles Hamiltonian to clarify how C_3-symmetry changes a topological feature of each bifurcation[1]. Bifurcation in the two dimensional Hamiltonian system is classified by Meyer for generic cases,[2] but symmetry of the system may cause a bifurcation, to which the classification can not be applied. It is still an open problem to determine the bifurcation type in symmetric systems.

It is observed in ref.1 that C_3-symmetric orbits can bifurcate three pairs of periodic orbits and the Poincar'e map shows the chain of 3k islands. In this note, details of the analysis of such bifurcations are reported. For more details of numerical results, see ref.1.

The normal form analysis for the bifurcation of a C_3-symmetric orbit.

The oscillations about a closed orbit in a system with two degrees of freedom are described by a time-periodic system with one degree of freedom. Following Ozorio,[3] let us consider the Fourier expansion of this Hamiltonian $h(p,q,t)$ close to the kth-order resonance ($k \geq 3$)

$$-2\mathrm{i}h(p(z,\bar{z}),q(z,\bar{z}),t) = -\mathrm{i}\omega z\bar{z} + \sum_{m+\bar{m}=3}^{\infty} \sum_{n=-\infty}^{\infty} h_{m\bar{m}n} z^m \bar{z}^{\bar{m}} \mathrm{e}^{\mathrm{i}nt}, \tag{1}$$

where $\omega = l/k + \epsilon$ and the rational l/k is the rotation number, ϵ ($|\epsilon| \ll 1$) is the bifurcation prameter. We set $l=1$ for the sake of simplicity. (p,q) are canonical variables and the variables z and $\bar{z}$ are defined by $z = p + \mathrm{i}q, \bar{z} = p - \mathrm{i}q$. As the Hamiltonian h is real, the coefficients satisfy $-\overline{h_{m\bar{m}n}} = h_{\bar{m}m-n}$.

In the Hamiltonian (1), the terms which satisfy $\omega(m-\bar{m}) - n = 0$ are called resonant terms. For the bifurcation analysis of the closed orbit, we need only the lowest order resonant terms, and not the trivial resonant terms $h_{kk0}(\bar{Z}Z)^k$. Up to the degree s_r, to which such lowest order resonant terms first appear, we can eliminate all non-resonant terms of the Hamiltonian h by use of succesive canonical transformations. The canonical transformation defined by the following generating function,

$$\begin{aligned} S(Z,\bar{z},t) &= Z\bar{z} + \sum_{m+\bar{m}=s} \sum_{n=-\infty}^{\infty} S_{m\bar{m}n} Z^m \bar{z}^{\bar{m}} \mathrm{e}^{-\mathrm{i}nt} \\ S_{m\bar{m}n} &= \frac{\mathrm{i}h_{m\bar{m}n}}{\omega(m-\bar{m})-n} \quad \text{for} \quad \omega(m-\bar{m})-n \neq 0 \\ &= 0 \qquad\qquad\qquad \text{for} \quad \omega(m-\bar{m})-n = 0, \end{aligned} \tag{2}$$

eliminates all s th-order non-resonant terms.

Here, we consider the threefold symmetric system such as the Hénon-Heiles potential. The Hamiltonian (1) of a C_3-symmetric orbit is invariant to the phase shift of $2\pi/3$, i.e.,

$$h_{m\bar{m}n} = 0 \quad \text{for} \quad n \not\equiv 0 \ (\mathit{mod}\ 3\,). \tag{3}$$

Here, we assume a certain generic property for the C_3-symmetric Hamiltonian (1) under the condition (3), namely, lowest order resonant terms do not vanish.[4] In this case, we obtain the

$s_r(=3k)$th-order normal form

$$-2\mathrm{i}h'(Z,\bar{Z},t) = -\mathrm{i}\omega Z\bar{Z} + \sum_{i=2}^{[3k/2]} h'_{ii0}(Z\bar{Z})^i + h'_{3k,0,3}Z^{3k}\mathrm{e}^{\mathrm{i}3t} + h'_{0,3k,-3}\bar{Z}^{3k}\mathrm{e}^{-\mathrm{i}3t} \tag{4}$$

by successive (3k-2) operations of the above canonical transformation.

Furthermore time dependence of h' can be eliminated and transformed into a new Hamiltonian h'', by the canonical transformation, of which the generating function σ is given by $\sigma = Z\bar{\xi}\mathrm{e}^{\mathrm{i}\frac{1}{k}t}$,

$$-2\mathrm{i}h''(\xi,\bar{\xi}) = -\mathrm{i}\epsilon\xi\bar{\xi} + \sum_{i=2}^{[3k/2]} h'_{ii0}(\xi\bar{\xi})^i + h'_{3k,0,3}\xi^{3k} + h'_{0,3k,-3}\bar{\xi}^{3k}. \tag{5}$$

Introducing the polar coordinates (I,ϕ) by the relation $\xi = (2I)^{1/2}\exp(\mathrm{i}\theta)$ and rotating the angle θ, we obtain the Hamiltonian describing the nonlinear effect for the k th-order resonance,

$$h(I,\phi:k) = \epsilon a_1 I + a_2 I^2 + \cdots + bI^{3k/2}\sin(3k\phi). \tag{6}$$

with the cancellation of linear time dependence in eq.(5).

Let us consider the bifurcation descrived by the system (6). The fixed points (I^*,ϕ^*) of the Hamiltonian (6) are given by equations $\dot{I}=0,\ \dot{\phi}=0$, i.e.,

$$\begin{aligned} \phi^* &= (n+1/2)\pi/3k \qquad (n=1,\cdots,6k) \\ -a_1\epsilon &= 2a_2I^* + 3a_3I^{*2} + \cdots. \end{aligned} \tag{7}$$

Therefore we have 6k fixed points after the bifurcation. Here, we take account of the linear part of the map $(I,\phi)\mapsto(I',\phi')$ corresponding to the Poincaré map. While the time t varies from 0 to 2π, the angle ϕ increases $2\pi/k$. Thus we have the relation

$$\phi' = \phi + 2\pi/k. \tag{8}$$

This implies that each branch consists of k fixed points and we have 6 individual closed orbits. Furthermore if parameters allow the domination of the oscillatory term in eq.(6), a half of these periodic orbits are elliptic and the others are hyperbolic.

To conclude, the bifurcation of a C_3-symmetric orbit branches off three pairs of orbits under the assumption above eq.(4). This result is completely consistent with numerical examples in ref.1.

1) S. Kurosaki: J.Phys.Soc.Jpn **64** (1995) 3589.

2) K. R. Meyer: Trans. AMS **149** (1970) 95, **154** (1971) 273.

3) A. M. Ozorio de Almeida: *Hamiltonian Systems: Chaos and Quantization* (Cambridge, New York, 1988) Part 1, Chap. 4, p. 91.

4) If the bifurcation process contains additional symmetry, this assumption or eq.(3) can be invalid. (For example, in the case of branches also possessing C_3-symmetry, the bifurcation consistent with Meyer's classification is obserbed.)